Expander Course - Homework set 3

Due: April 15

Problem 1. Let A be a polynomial time probabilistic algorithm which flips k random vectors of m bits each, $X_1, \cdots, X_k$, and rejects iff $X_i \in A_i$ for at least $k/2$ is, where the A_i are unknown fixed subsets of $\{0,1\}^m$, each of size at most $2^m/500$. Clearly the rejection probability is at most 2^{-k}. Devise another efficient algorithm B which has a similar rejection probability, but uses at most $m + O(k)$ random bits.
Hint: generalize the argument given in class to handle the probability of an expander walk staying inside different sets in different steps of the walk. Explain how you choose the parameters of the expander for the error and efficiency analysis of your algorithm.

Problem 2. Throughout H will be a finite Abelian group (written additively). Recall that every Abelian group is a direct sum of cyclic groups. Denote by Z_m the cyclic group of size m, with elements $\{0, 1, , \cdots, m-1\}$. In the next problems you'll figure out the Fourier transform of Abelian groups, and apply them to finding the spectrum of Cayley graphs (defined by groups).

The *dual group* $\hat{H}$ is the set of all functions (called characters of H) χ from H to the unit circle (complex numbers of absolute value 1), satisfying for all $g, h \in H$ $\chi(g+h) = \chi(g)\chi(h)$.

1. Let ω be a primitive mth root of unity. Prove that for every $a \in Z_m$ the function χ_a defined by $\chi_a(b) = \omega^{ab}$ is a character of Z_m. Conclude that $\hat{Z}_m$ is a group, which is isomorphic to Z_m. Also note that every pair of these functions is orthogonal (under the usual complex inner product).

2. Show that χ is a character of H_1 and ψ is a character of H_2 then their product $\chi\psi$ is a character of the direct sum $H_1 + H_2$, and that all characters of this direct sum arise this way. Conclude that this allows the construction of $\hat{H}$ for every Abelian H.

3. Write explicitly the characters of Z_m^n.

Problem 3. Let us fix an Abelian group H.

1. Consider the (Fourier transform) matrix M whose rows are indexed by elements of H and columns by elements by the of $\hat{H}$, defined by $M(h, \chi) = \chi(h)$. Prove that all its rows are pairwise orthogonal, and so are the columns.

2. Let f be a complex function on H, and let F be a matrix with rows and columns indexed by H, defined by $F(g, h) = f(g - h)$. Prove that the columns of M are the complete set of eigenvectors of F (and so the Fourier transform diagonalizes F).

Problem 4. In this problem we will sometimes consider (families of) graphs of unbounded degree. Still, the definition of an expander family makes sense - $\{G_i\}$ which are d_i-regular are an expander family if for some $a < 1$ and all i, $\lambda(G_i) < ad_i$.

Let S be a set of generators for H, and let the Cayley graph $C(H; S)$ be defined with vertices H and edges (g, h) for all such (ordered) pairs satisfying $g - h \in S$. Note that this is a directed graph, but we will usually (not always) deal with the case $S = -S$ in which the graph becomes undirected. (If S is a multiset the graph may have parallel edges too). Note that the adjacency matrix of C is diagonalizable by M above.

1. Determine all eigenvalues of $C(Z_2^d; S)$ with S being all standard unit vectors e_i for $1 \leq i \leq d$ (this is the graph of the Boolean cube). Conclude that this is a bad expander family, even though the degree is nonconstant. Can you find a small cut?

2. Show that if S in the graph above is a random set of $2d$ Boolean d-vectors , then with high probability the graph is a good expander family.

3. Determine all eigenvalues of $C(Z_n; S)$ with $S = \{1\}$ and with $S = \{-1, 1\}$. (these are the graphs of the directed and undirected cycle). Conclude that they are bad expanders.

4. Show that if $|S| = o(\log n)$ then the graph above cannot be a good expander.

5. Show that if S is chosen at random with $|S| >> \log n$ then with high probability it is an expander.